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LETTER TO THE EDITOR

Some interesting charmonium potentials

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Abstract. A simple technique is used to extend and clarify recent results on some charmonium confining potentials. One such potential has a convergent Rayleigh–Schrödinger series which does not give the correct bound-state energy.

In a recent note Datta and Mukherjee (1980) studied a Schrödinger equation with a confining potential term of the type used in models of the charmonium system:

$$-\nabla^2\psi - \frac{\zeta}{r}\psi + (\alpha^2 r + \alpha^4 r^2)\psi = E\psi. \tag{1}$$

They derived the scaling result

$$E(\zeta, \alpha) = \alpha^2 E(\zeta\alpha^{-1}, 1) \tag{2}$$

and concluded that $E(\zeta, \alpha)$ for fixed real ζ has a convergent expansion in α^{-1} for sufficiently large α. In this Letter we wish to point out that the relevant series converges because it is *finite* for the ground state of each angular momentum type, and we conjecture that the same holds for the excited states. We also wish to emphasise that, for various simple perturbation problems, the easiest way to proceed is to use the method pointed out by Killingbeck (1977) i.e. write down the eigenfunction *first*, and then evaluate the potential. A solid harmonic factor y_l in the function ensures that it has angular momentum l, and a radial factor $e^{-f(r)}$ with $f(r) \to \infty$ as $r \to \infty$ will ensure that the function can be normalised. If we set $\psi_l = y_l e^{-f(r)}$ we quickly find

$$-\nabla^2\psi_l = [f'' - (f')^2 + (2l+2)r^{-1}f']\psi_l. \tag{3}$$

Since $e^{-f(r)}$ will give no radial nodes, we are dealing with ground states for each l, but an extra factor in ψ_l with radial nodes would presumably allow excited states to be treated as well. The choice $f = \frac{1}{2}r + \frac{1}{2}\alpha^2 r^2$ gives

$$[-\nabla^2 - (l+1)r^{-1} + \alpha^2 r + \alpha^4 r^2]\psi_l = [-\tfrac{1}{4} + (2l+3)\alpha^2]\psi_l. \tag{4}$$

Following the form of equation (2), we can write the energy as

$$E = \alpha^2(E_0 + E_1\alpha^{-1} + E_2\alpha^{-2} + \ldots) \tag{5}$$

with only E_0 and E_2 non-zero, which is the point which was to be established. By making the choice $f = \frac{1}{2}r + g(r)$, where $g(r)$ is a polynomial in r, a variety of charmonium-type potentials could be found which also yield finite energy series. For the case studied above both the α series and the α^{-1} series for the energy are finite. However if the term $\alpha^2 r$ is replaced by $K\alpha^2 r$ in (4), with K deviating by even a little from 1, the α series for

the energy becomes divergent asymptotic. Even at $K=1$ the α series for quantities such as $\langle r\rangle$ is divergent asymptotic, so that the behaviour of the energy stands out as exceptional. Whether the situation is the same for the α^{-1} series apparently still remains to be investigated.

A confining potential based on an isotropic oscillator potential, with no gluon $1/r$ term, results if we set $f=\frac{1}{2}\alpha r^2+\beta r^4$ in (3). We find

$$[-\nabla^2+\omega^2 r^2+\tfrac{1}{2}\lambda r^4+\tfrac{1}{3}\eta r^6]\psi_l=(2l+3)\alpha\psi_l \tag{6}$$

where

$$\lambda=16\alpha\beta, \qquad \eta=48\beta^2, \qquad \omega^2=\alpha^2-(8l+20)\beta.$$

Flessas and Das (1980) have studied this case, starting from the Schrödinger equation and deducing ψ_l after a lengthy calculation. Our alternative procedure is very simple and less liable to error; we note that the result

$$(3\lambda^2/16\eta)=\omega^2+(8l+20)(\eta/48)^{1/2}, \tag{7}$$

which follows from our calculation, differs from the result for $(3\lambda^2/16\eta)$ given in equation (17) of Flessas and Das (1980). If $\omega=1$ and β is small, then a term $-16\beta^2 r^2$ can be treated as a small perturbation (with ψ_l the exact *unperturbed* eigenfunction) if we wish to know the energy for an oscillator perturbed by an r^4 term alone (Killingbeck 1978a). With an *exact* unperturbed function, the method of Aharonov and Au (1979) provides quadrature formulae for the coefficients in the series for the small energy correction.

If the parameter $\lambda=\alpha^2$ in equation (4) is taken to be real and negative then (4) is still formally correct for finite r, but the corresponding ψ_l is not square-integrable. Because of the r^4 term, the potential on the left of (4) has bound states *both* for $\lambda>0$ and for $\lambda<0$, but only for $\lambda>0$ does the perturbation series $-\frac{1}{4}+(2l+3)\lambda$ give the exact energy. For $\lambda<0$ the quantity $\Delta=E(\lambda)+\frac{1}{4}-(2l+3)\lambda$ is positive and its magnitude increases with $|\lambda|$. The table shows a few typical results; the exact eigenvalues were found by numerical integration. This phenomenon is different from the usual one (exemplified by the potential $-r^{-1}+\lambda r$) in which there are bound states for $\lambda>0$ (but not for $\lambda<0$) and the energy perturbation series is divergent asymptotic. We anticipate that the quantity Δ tabulated in table 1 represents some function of λ which is non-analytic at $\lambda=0$, and so is not represented by the series. For $\lambda>0$, Δ is zero, and the numerical integration method which we used was tested by confirming that it correctly gave this result to within 10^{-8}. For $\lambda<0$ the value of Δ tends to zero rapidly with $|\lambda|$; for the 1s state, $\Delta(-0{\cdot}01)$ is less than 10^{-8} and so could only be found roughly by our single precision integration. The finite series for the energy on the right of equation (4) is, of course, exact (for $\lambda>0$), and we intuitively expect that it is what we would obtain by formally calculating the Rayleigh–Schrödinger perturbation series for

Table 1. Δ values for the Hamiltonian of equation (4).

λ	$l=0$	$l=1$
−0·02	0·000 059 10	0·001 203 07
−0·03	0·001 586 94	0·011 756 18
−0·04	0·037 106 08	0·043 506 79

a hydrogenic atom with perturbation term $\lambda r + \lambda^2 r^2$. Explicit analytic evaluation of E_2 and E_3 shows that they are indeed zero; to proceed further we have used the hypervirial method of Killingbeck (1978b) in a double precision computer calculation to show explicitly that the $E_n (2 \leqslant n \leqslant 16)$ are zero. A single precision calculation only gives *exactly* zero for the E_n with $2 \leqslant n \leqslant 10$, since this particular case is a severe test of the method; each term (λr and $\lambda^2 r^2$) on its own gives a viciously divergent energy series, but their sum gives a 'destructive interference' effect which yields $E_n = 0$ for $n > 1$. This behaviour is, of course, highly unstable with respect to a change in the coefficient of the r term. This isolated change from a divergent energy series to a convergent one is really rather misleading, since the *exact* eigenvalue changes smoothly as the coefficient of the r term varies. However, the singular case does give an exact eigenfunction which may be useful mathematically, if used in conjunction with the method of Aharonov and Au (1979), as explained in our preceding comments on the oscillator problem.

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